

# A mathematical model of the Mafia game

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Mafia (also called Werewolf) is a party game. The participants are divided into two competing groups: citizens and a mafia. The objective is to eliminate the opponent group. The game consists of two consecutive phases (day and night) and a certain set of actions (e.g. lynching during day). The mafia members have additional powers (knowing each other, killing during night) whereas the citizens are more numerous.

We propose a simple mathematical model of the game, which is essentially a pure death process with discrete time. We find the closed-form solutions for the mafia winning-chance  $w(n, m)$  as well as for the evolution of the game. Moreover, we investigate the discrete properties of results, as well as their continuous-time approximations.

It turns out that a relatively small number of the mafia members, i.e. proportional to the square root of the total number of players, gives equal winning-chance for both groups. Furthermore, the game strongly depends on the parity of the total number of players.

## I. INTRODUCTION

Mafia (also called Werewolf) is a popular party game [1]. The participants collect in a circle and a game coordinator assigns each player to one of two groups: a mafia or citizens. Citizens know only their own identity, whereas mafia members know identity of their fellows. The objective is to eliminate the opponent group. The game consists of two alternating phases (day and night). During the day, all players can discuss and vote who they want to lynch. During the night the mafia kills a citizen of their choice.

There are many variants of the mafia game. The most common modification of the gameplay is the addition of characters with special abilities. Usually there are two special citizens: Detective (who checks every night if a chosen person is in the mafia) and Nurse (who may protect a victim from being killed, if she chooses correctly).

In this paper we analyze the simplest version of the Mafia game. That is, with only mafia and citizens, and without any special players or additional rules. Even if it may not be the most popular variant, it is the best one for mathematical modelling. Despite the game having a complex psychological component it is possible to create a stochastic model of the Mafia game. We consider a game beginning with  $n$  players, out of which there are  $m$  mafia members. The main questions we address are:

- What is the probability that the mafia wins  $w(n, m)$ ?
- How does the dynamics of the game look like? That is, what is the chance that after a given time there is exactly a certain number of mafia members?

Besides the direct answers (i.e. closed form expressions), we study approximations, qualitative behavior and some special cases.

There are only a few previous papers on the Mafia game. Works [2] and [3] basically propose the same model as the one presented in this paper. They calculate a simple asymptotic formula for the mafia-winning chance,  $w(n, m) \propto m/\sqrt{n}$ .

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Other research projects concentrate on the psychological aspect of the Mafia game, in particular — deceiving. Topics analyzed include: patterns of a deceiver’s interruptions and voice parameters [4], movement of face and hands [5] and usage of language [6]. Also, the Mafia game is applied as a test for some videoconferencing setups [7].

The research is motivated by two goals, besides the sheer fun of calculation and writing. First, a good theoretical model may be useful for the investigation of the psychological aspect of the Mafia game. Experimental deviations from the idealized behavior may give a valuable insight into psychology of strategy choosing, manipulation, deceiving, following others and hiding identity. Second, when a small, but well-informed and powerful group fights against the majority, the nature of the process may be similar to that of the Mafia game. Such real-life phenomena include actions of secret societies as in high-stake corruption, terrorism and illegal oppositions.

The paper is organized as follows. In Section II we introduce the rules of the Mafia game and propose a simple mathematical model. Section III contains a special case of play with only a single mafia member. The results are not only simple and didactic, but also useful for a more general case. Section IV gives simple results on the qualitative behavior of the mafia-winning chance. Section V presents the dynamical aspect of the Mafia game and gives closed-form expressions for dynamics of the game. We consider both the discrete-time model and its continuous time approximation. In Section VI we provide the exact result for the mafia-winning chance, together with its asymptotic approximation. Section VII concludes the work and gives insight into possible extensions and applications. Appendices A and B contain derivations of formulas found in Sec. V.

## II. MODEL

Before proceeding to mathematics, we need to write down our arbitrarily chosen rules of the Mafia game. Some assumptions are made to simplify the model, others purely for convenience. Note that this Section is only a hand-waving transition from the real-life game to its mathematical model.

- The game needs  $n$  players and one more person to coordinate it.
- At the beginning players are randomly divided into  $(n - m)$  citizens and  $m$  mafia members.
- Mafia members know the identity of each other, citizens — only their own.
- There are two alternating phases, day and night, which together comprise a turn.
- During day, there are two consecutive subphases:
  - Debate. Everyone still alive can say anything related to accusing or defending.
  - Voting. Everyone has exactly one vote, who he wants to lynch. The player who gets the highest number of votes is eliminated (in case of a tie, a random ‘winner’ is eliminated).
- During the night:
  - Mafia jointly decides who they want to kill (eliminate).
- The game continues until there is only one group (either the citizens or the mafia) left. That group wins.

Stated briefly, during the day one player is eliminated (either a citizen or a mafia member) whereas during the night the mafia kills one citizen. Consequently, during a single turn the possible transitions are:  $(n, m) \rightarrow (n - 2, m)$  and  $(n, m) \rightarrow (n - 2, m - 1)$ . Their probabilities, in principle, depend on  $n$ ,  $m$  and the course of the play (i.e. previous discussions and votings). To start with, we need to get rid of the psychological aspect, restricting ourselves only to strategic and probabilistic parts of the Mafia game. It is a coarse approximation, as psychology plays an essential role in the Mafia game. Nevertheless, even such a bare model has interesting properties. Moreover, we want to neglect the discussion phase as it is extremely difficult to formalize in a meaningful and useful way.

Now, let's make a hand-waving argument why the transition probability depends only on the current state  $(n, m)$ . Consider the scenario, in which players are lynched at random, that is

$$P[(n-2, m)|(n, m)] = \frac{n-m}{n} \quad \text{and} \quad P[(n-2, m-1)|(n, m)] = \frac{m}{n}. \quad (1)$$

Let's denote by  $w(n, m)$  the mafia-winning chance, induced by the above transition probabilities.

Suppose that there is a better strategy (in the common sense of the word) for citizens, giving it  $w'(n, m) < w(n, m)$ . Then the mafia can force a random lynch, just by pretending they are citizens. It is possible as:

- Each mafia member knows more than any citizen.
- In such case there is no way to detect a mafia member.

Suppose that there is a better strategy for the mafia,  $w''(n, m) > w(n, m)$ . If the citizens are in majority, they can force a random lynching by the following procedure:

- One citizen says "let's every of us give a number  $k_i$  from 1 to  $n$ , we will kill the  $\sum k_i \bmod n$ -th".
- The number  $\sum k_i \bmod n$  is random, so is the chosen player.
- As there are more citizens than mafia members, mafia votes do not play any role.

If the citizens are in the minority (i.e.  $m > n - m$ ) no matter what the moves are, the mafia will win. For  $m = n - m$  let's assume a random player is lynched.

Consequently, random lynch is a well-justified strategy. A more detailed argument is shown in [2]. As a direct consequence of (1), the mafia winning-chance can be expressed as a recurrence equation

$$w(n, m) = \begin{cases} 0 & \text{if } m = 0, \\ 1 & \text{if } m > n - m, \\ \frac{n-m}{n}w(n-2, m) + \frac{m}{n}w(n-2, m-1) & \text{in all other cases.} \end{cases} \quad (2)$$

The above equation is the core of our paper. We investigate it for one mafia member (Section III), its qualitative behavior (Section IV) and closed-form solution along with its asymptotic formula (Section VI).

It is important to point out that we may set different boundary conditions, i.e. in which if there is the same number of citizens and mafia members, mafia wins. That is, in (2) we may write (1 if  $m \geq n - m$ ) instead of (1 if  $m > n - m$ ). In that case all results can be reproduced, perhaps in a slightly changed form (i.e. different constants and even $\leftrightarrow$ odd).

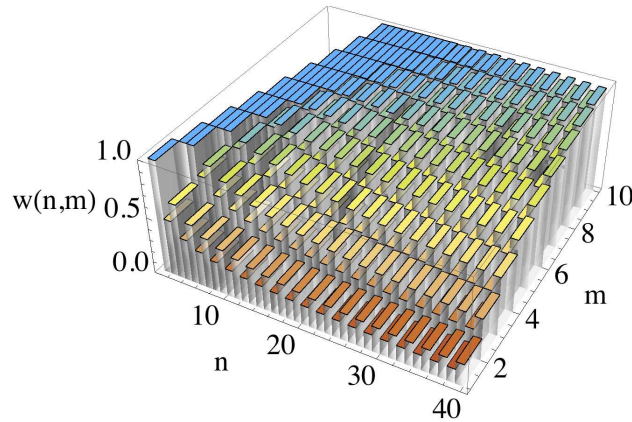


Figure 1: A numerical plot of the probability that the mafia wins  $w(n, m)$ .

### III. GAME WITH ONE MAFIA MEMBER

To start with, we want to consider a very simple case — the game with only a single mafia member. That is, in this Section we analyze  $w(n, 1)$ . Besides the simplicity, there are two more motivations: this case gives insight into some general properties and the results will be useful in the further part of the paper.

In a game with one mafia member, their winning probability is the chance that during every lynch a citizen is killed. Bearing in mind that we start the game with the day phase, we get

$$w(n, 1) = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \dots \cdot \frac{1+(n \bmod 2)}{2+(n \bmod 2)} = \frac{(n-1)!!}{n!!}. \quad (3)$$

The above formula has an explicit dependence on the parity of  $n$ . Even though it is obvious that addition of 2 citizens reduces mafia-winning chance  $w(n+2, 1) < w(n, 1)$ , it is not the case for addition of a single citizen. In fact, addition of an odd player (holding  $m = 1$ ) increases the mafia-winning chance. To understand this let's use an example. In a play with one citizen and one mafia member, there is tie in voting, so lynching relies on tossing a coin and  $w(2, 1) = 1/2$ . In a game with two citizen, the mafia also wins after lynching a citizen (the second one is to be killed during the night). This time it is more difficult to hunt mafia, and  $w(3, 1) = 2/3$ . One may think that we encountered an issue of boundary conditions (i.e. killing a random player in a tie) or that we considered too small  $n$ . Neither is the case. A careful reader may easily check that changing (1 if  $m > n - m$ ) to (1 if  $m \geq n - m$ ) in (2) we get an analogous phenomena.

We may simply check that adding an odd player always increases the mafia winning-chance,

$$\begin{aligned} w(2k+1, 1) - w(2k, 1) &= \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdot \dots \cdot \frac{4}{5} \cdot \frac{2}{3} + \\ &\quad - \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdot \dots \cdot \frac{3}{4} \cdot \frac{1}{2} \\ &> \frac{\sqrt{(2k+1)(2k-1)}}{2k+1} \cdot \frac{\sqrt{(2k-1)(2k-3)}}{2k-1} \cdot \dots \cdot \frac{\sqrt{5 \cdot 3}}{5} \cdot \frac{\sqrt{3 \cdot 1}}{3} + \\ &\quad - \frac{2k-1}{\sqrt{(2k+1)(2k-1)}} \cdot \frac{2k-3}{\sqrt{(2k-1)(2k-3)}} \cdot \dots \cdot \frac{3}{\sqrt{5 \cdot 3}} \cdot \frac{1}{\sqrt{3 \cdot 1}} = 0, \end{aligned} \quad (4)$$

that is,  $w(2k+1, 1) > w(2k, 1)$ .

Let's consider  $w(n, 1)$  averaged over neighbouring numbers. We use the geometric mean, as it simplifies the result

$$\sqrt{w(n-1, 1)w(n, 1)} = \sqrt{\frac{(n-2)!!}{(n-1)!!} \frac{(n-1)!!}{n!!}} = \frac{1}{\sqrt{n}}. \quad (5)$$

The averaging may be considered an approximation in which we get a monotonic function (without the sawtooth pattern). To get some information on the dependence of  $w(n, 1)$  on the parity of  $n$  we consider the following ratio

$$\frac{w(2k+1, 1)}{w(2k, 1)} = \frac{(2k)!!}{(2k+1)!!} / \frac{(2k-1)!!}{(2k)!!} = \left( \frac{(2k)!!}{(2k-1)!!} \right)^2 \frac{1}{2k+1}. \quad (6)$$

The above expression has a limit when  $k$  goes to infinity, which can be found with the aid of the Wallis formula:

$$\frac{\pi}{2} = \lim_{k \rightarrow \infty} \left( \frac{(2k)!!}{(2k-1)!!} \right)^2 \frac{1}{2k+1}. \quad (7)$$

We may develop approximate formulas for the single mafia member winning-chance, which take into account

parity of the number of players

$$\begin{aligned}
 w(2k, 1) &= \left( \frac{w(2k+1, 1)}{w(2k, 1)} \right)^{-\frac{1}{2}} \sqrt{w(2k, 1)w(2k+1, 1)} \\
 &\approx \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2k+1}}, \\
 w(2k+1, 1) &= \left( \frac{w(2k+1, 1)}{w(2k, 1)} \right)^{\frac{1}{2}} \sqrt{w(2k, 1)w(2k+1, 1)} \\
 &\approx \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2k+1}}.
 \end{aligned} \tag{8}$$

Or, bearing in mind  $(n+1)/n \rightarrow 1$ ,

$$w(n, 1) \approx \left( \frac{\pi}{2} \right)^{(n \bmod 2) - 1/2} \frac{1}{\sqrt{n}}. \tag{9}$$

As we see, the recurrence equation produces results that might be counter-intuitive. Imagine there are four of us and we are playing with a single mafia member. His winning-chance is  $w(4, 1) = \frac{3}{8} = 0.375$ . Then we invite five more friends to play as additional citizens. Consequently, the mafia winning-chance rise to  $w(9, 1) = \frac{128}{315} \approx 0.406$  (contrary to a naive expectation).

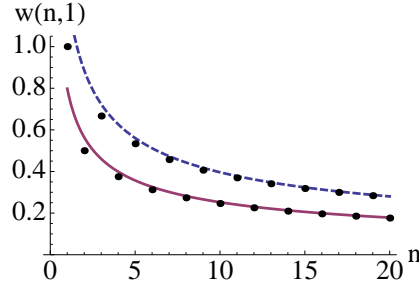


Figure 2: Plot of the mafia winning-chance for a game with a single mafia member, that is  $w(n, 1)$ . Dots illustrate the exact result (3), whereas the lines represent the approximations (9) — solid and dashed line for even and odd number of players, respectively.

#### IV. QUALITATIVE PROPERTIES OF $w(n, m)$

It is worth investigating the qualitative behavior of  $w(n, m)$ . Some questions, which naturally arise, are:

- Does the addition of two citizens increase their winning-chance, i.e. is  $w(n+2, m) < w(n, m)$ ?
- Does the addition of one citizen and one mafia member increase the mafia winning-chance, i.e. is  $w(n+1, m+1) > w(n, m)$ ?
- Does the change of one player from a mafia member to a citizen decrease the mafia winning-chance, i.e. is  $w(n, m+1) > w(n, m)$ ?

In fact, the above questions are equivalent, as we can see using the recurrence relation (2)

$$w(n, m) = \frac{n-m}{n} w(n-2, m) + \frac{m}{n} w(n-2, m-1)$$

for  $n - m \geq m > 0$ . After some cosmetic arithmetical operations we get

$$\begin{aligned} (n - m) [w(n - 2, m) - w(n, m)] &= m [w(n, m) - w(n, m - 1)], \\ n [w(n, m) - w(n - 2, m - 1)] &= (n - m) [w(n - 2, m) - w(n - 2, m - 1)]. \end{aligned} \quad (10)$$

The above relations can be directly translated into

$$w(n - 2, m) > w(n, m) \Leftrightarrow w(n, m) > w(n - 2, m - 1) \Leftrightarrow w(n - 2, m) > w(n - 2, m - 1). \quad (11)$$

In addition to the equivalence of the 'obvious' inequalities, we are interested whether they are fulfilled. Let's utilize the third part of (11) in the following form:

If a mafia member is changed into a citizen, the mafia winning-chance does not increase, that is  $w(n + 1, m + 1) \geq w(n + 1, m)$ . If additionally  $n - m \geq m$ , the inequality is strong,  $w(n + 1, m + 1) > w(n + 1, m)$ .

Let's introduce an agent to a game with  $n$  players among which there are  $m$  mafia members. The agent is an additional player who simulates either a citizen or a mafia member. We are going to show that the agent does not have to reveal its identity until the very late part of the game.

As citizens are indistinguishable, let's assume that mafia kills them during night in a fixed sequence. Let the agent have the last position. During the lynching there are three possibilities

- A citizen dies.
- A mafia member dies.
- The agent dies.

In the case of the agent's death, his affiliation has no meaning. In the two remaining situations, the game continues with the agent. The agent's affiliation plays a decisive role in two situations:

- When all mafia members are dead.
- When 0–1 citizens and 1–2 mafia members remain.

In both cases when the agent is a mafia member, the mafia winning-chance increase:

Game state $(n, m)$	$w$ if the agent is of mafia	$w$ if the agent is of citizens
$(n, 0)$	0	$w(n, 1)$
$(1, 1)$	$\frac{1}{2}$	1
$(2, 1)$	$\frac{1}{3}$	1
$(3, 2)$	$\frac{1}{4}$	1

The total mafia winning-chance is equal to the mean winning-chance, averaged over disjoint games. The weight is the probability of achieving a such game. The change of the agent's affiliation does not affect the weights, but in the ending always increases the mafia winning-chance,  $w(n + 1, m + 1) \geq w(n + 1, m)$ . Additionally, for  $n - m \geq m \geq 1$  there is a game in which all mafia members are killed and the agent's affiliation plays a role, hence  $w(n + 1, m + 1) > w(n + 1, m)$ .

Let's show one more property — when  $n - m \geq m$  adding an odd player always increases the mafia winning-chance, not only in games with one mafia member (4). We will use mathematical induction, using the boundary conditions (2) and the already shown properties of  $w(n, 1)$  as the basis. The inductive assumption is that the property holds for the total number of players  $n \leq 2k + 1$ , and the inductive step is the following:

$$\begin{aligned} w(2(k + 1) + 1, m) &= \frac{2k + 3 - m}{2k + 3} w(2k + 1, m) + \frac{m}{2k + 3} w(2k + 1, m - 1) \\ &> \frac{2k + 3 - m}{2k + 3} w(2k, m) + \frac{m}{2k + 3} w(2k, m - 1) \\ &> \frac{2k + 2 - m}{2k + 2} w(2k, m) + \frac{m}{2k + 2} w(2k, m - 1) \\ &= w(2(k + 1), m), \end{aligned} \quad (12)$$

where we subsequently used the inductive assumption and made use of the fact  $w(2k, m) > w(2k, m - 1)$ .

## V. EVOLUTION

In the two previous Sections we considered the mafia winning-chance with respect to the initial number of players ( $N$ ) and mafia members ( $M$ ). It may be interesting to explore dynamics of the play, that is, analyze the probability  $p_m(t)$  that after the  $t$ -th turn there will be exactly  $m$  mafia members. Even though this goes beyond the (2), it is a direct consequence of the random lynching model (1). Throughout this section we use capital letters  $N$  and  $M$  to denote the initial conditions.

Note that every turn, or day and night, two players are killed, so the total number of player decreases  $n(t) = N - 2t$ .

### A. Discrete time

The evolution of each  $p_m(t)$  is governed by a set of recurrent equations

$$p_m(t+1) = \frac{(N-2t)-m}{N-2t} p_m(t) + \frac{m+1}{N-2t} p_{m+1}(t), \quad (13)$$

where  $m \in \mathbb{N}$  and with the initial condition  $p_m(0) = \delta_{mM}$ .

Note that the above is a stochastic process with  $M+1$  distinct states and  $(m) \rightarrow (m-1)$ . In general, such a Markov process is called a pure death process (a subclass of birth and death processes). In our case the transition probabilities change with time, which is a slight complication.

It is essential to point that the equations in (13) are correct only when  $m \leq n(t)$ , that is, when there is at least one citizen alive and the game can be still played. Otherwise they have no meaning. But why does an erroneous equation for  $m > n(t)$  not spoil states with  $m \leq n(t)$ ? For a given time  $t$  let  $p_m(t)$  be the erroneous probability with the smallest index, i.e.  $m = N - 2t + 1$ . In the next turn it affects  $p_{m-1}(t+1)$ . But, now the smallest wrong index is  $N - 2(t+1) + 1 = m - 2 < m - 1$ . So the error propagates more slowly than the exclusion of the states.

The closed form solution for  $p_m(t)$  reads

$$p_m(t) = \sum_{i=m}^M \binom{M}{i} \binom{i}{m} (-1)^{i-m} \frac{(N-2t)!!}{N!!} \frac{(N-i)!!}{(N-2t-i)!!} \quad (14)$$

and its derivation is presented in Appendix A. An example is plotted in Fig. 3 (a).

However, the expression above, although exact, is fairly complicated. Let's analyze the mean number of mafia players after  $t$  days and nights, that is

$$\langle m \rangle(t) = \sum_{m=0}^M m p_m(t). \quad (15)$$

In birth and death processes with transition coefficients proportional to state's label (as radioactive decay process or Yule process) such mean may be obtained by a straightforward calculation with the use of (13)

$$\langle m \rangle(t) = \frac{N-2t+1}{N-2t+2} \langle m \rangle(t-1) = \left( \prod_{i=0}^{t-1} \frac{N-2i-1}{N-2i} \right) M. \quad (16)$$

The result is correct only when  $N - 2t - M \geq 0$ , that is, the average is taken over probabilities of the correct states.

### B. Continuous time approximation

Usually differential equations are simpler than difference equations. In this Section we crudely approximate the discrete-time evolution (13) by its continuous-time version. We know that

$$p_m(t+1) - p_m(t) \xleftarrow{\Delta=1} \frac{p_m(t+\Delta) - p_m(t)}{\Delta} \xrightarrow{\Delta \rightarrow 0} \frac{d}{dt} p_m(t). \quad (17)$$

Let's change in (13) the difference  $p_m(t+1) - p_m(t)$  into the differential  $\frac{d}{dt} \tilde{p}_m(t)$

$$\frac{d}{dt} \tilde{p}_m(t) = -\frac{m}{N-2t} \tilde{p}_m(t) + \frac{m+1}{N-2t} \tilde{p}_{m+1}(t) \quad (18)$$

where  $m \in \mathbb{N}$  and with the initial condition  $\tilde{p}_m(0) = \delta_{mM}$ . When a relative change of the function derivative is over unit length interval  $[t, t+1]$ , the approximation should be good  $\tilde{p}_m(t) \approx p_m(t)$ . Intuitively speaking, when the chance of lynching a mafia member during one day is small, such an approximation is justified. However, in the course of this paper we do not estimate the approximation error.

The solution of the differential equation (18) reads

$$\tilde{p}_m(t) = \binom{M}{m} \left(1 - \sqrt{1 - \frac{2t}{N}}\right)^{M-m} \left(\sqrt{1 - \frac{2t}{N}}\right)^m, \quad (19)$$

see Appendix B for detailed derivation. This is more convenient to work with than the exact expression (14). An example is plotted in Fig. 3 (a).

For example, maxima of  $\tilde{p}_m(t)$  are easily found

$$t_m = \frac{N}{2} \left(1 - \left(\frac{m}{M}\right)^2\right). \quad (20)$$

The mean number of mafia players, defined as in (15), is

$$\langle \tilde{m} \rangle(t) = M \sqrt{1 - \frac{2t}{N}}, \quad (21)$$

which is a simpler expression than its analogue (16). The comparison of the discrete-time formula with its continuous-time approximation is in Fig. 3 (b).

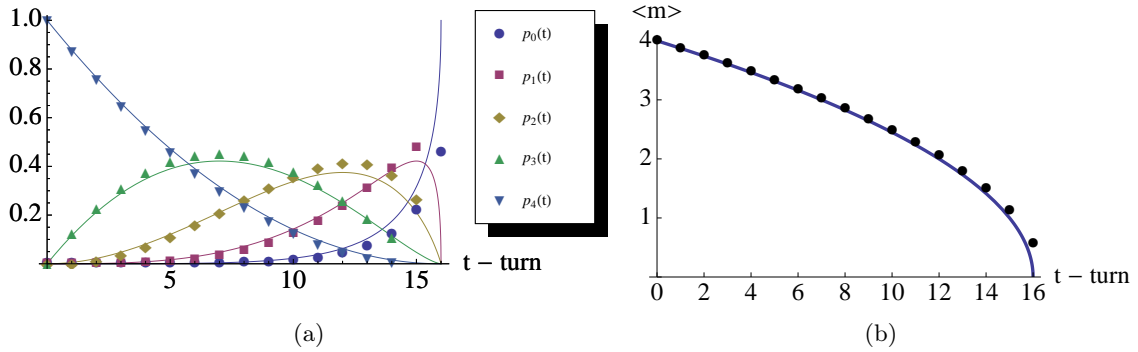


Figure 3: Dynamics of the Mafia game for the initial conditions  $N = 32$  and  $M = 4$ . We compare results for the discrete-time evolution with its continuous-time counterparts. (a) The probability that after  $t$  turns there are exactly  $m$  mafia members alive — the exact results  $p_m(t)$  (dots) and their approximations  $\tilde{p}_m(t)$  (lines). (b) The mean number mafia members — the discrete result  $\langle m \rangle(t)$  (dots) and its continuous time variant  $\langle \tilde{m} \rangle(t)$  (lines).



It is tempting to estimate the mafia winning-chance. As  $\tilde{p}_0(t)$  is the probability that the citizens won, setting time for which there is only one player left gives the total citizen winning-chance:

$$\begin{aligned}\tilde{w}(n, m) &= 1 - \tilde{p}_0\left(\frac{n-1}{2}\right) = 1 - \left(1 - \sqrt{1 - \frac{n-1}{n}}\right)^m \\ &\approx \frac{m}{\sqrt{n}}.\end{aligned}\tag{22}$$

Not surprisingly, in the continuous-time approximation, there is no explicit dependence on parity (the discrete properties are lost).

## VI. GENERAL SOLUTION

The result for  $p_m(t)$  (14) is not only interesting by itself — it also gives us the solution for the mafia winning-chance (2). Note that  $p_0(t)$  is the probability that after  $t$  turns citizens win. When total number of players is odd, game lasts for  $t_{\text{odd}} = (N-1)/2$  days and nights, leaving 1 player alive. Thus  $p_0(t_{\text{odd}})$  is the probability the citizen win the game. For an even number of players, after  $t_{\text{even}} = N/2$  turns there are no players alive. But now  $p_0(t_{\text{even}})$  is the probability that there are no mafia members, so in fact — no-one was killed during the last night and citizen win. So  $w(n, m) = 1 - p_0(\frac{N-N \bmod 2}{2})$ . Consequently, the closed form formula for the mafia winning-chance reads

$$w(n, m) = 1 - \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{(n-i)!!}{n!!((n \bmod 2) - i)!!}.\tag{23}$$

Let's show what we have proven for a game with only one mafia member — the asymptotic behavior (9).

To get the asymptotic formula for  $w(n, m)$  it suffices to notice that in the limit  $n \rightarrow \infty$  (taken over a selected parity) only the first two terms of the sum contribute. For  $i = 0$  the term is equal to 1. The terms with  $i > 1$  do not matter as  $(n-i)!!/(n-1)!! \rightarrow 0$ . Consequently, we may write

$$\begin{aligned}w(n, m) &\approx m \frac{(n-1)!!}{n!!} \\ &\approx \left(\frac{\pi}{2}\right)^{(n \bmod 2) - 1/2} \frac{m}{\sqrt{n}},\end{aligned}\tag{24}$$

where we used the result from game with one mafia member, that is, approximating (3) with (9). The approximate formula (24) can be formally expressed as  $\frac{w(n, m)}{m/\sqrt{m}}$

It is tempting to ask for the optimal number of mafia members  $m_{\text{opt}}$ , for a given number of players. An interesting game is one in which both groups have the same chance of winning

$$m_{\text{opt}} \approx \frac{1}{2} \left(\frac{\pi}{2}\right)^{-(n \bmod 2) + 1/2} \sqrt{n},\tag{25}$$

which are plotted in Fig. 4.

## VII. CONCLUSION

In this paper we make, and solve, a mathematical model of the Mafia game. The starting point of our calculations is random lynching process (1).

The main interest of this research is analysis of the mafia winning-chance  $w(n, m)$  (2). We find the closed-form solution (23), as well as its asymptotic approximation (24). We prove some discrete properties of

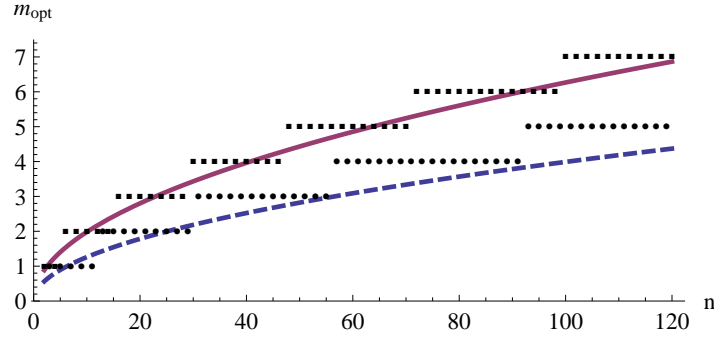


Figure 4: Optimal number of mafia members for a given number of players. Points show numerical results, that is, number of mafia members  $m_{opt}$  for which  $w(n, m_{opt})$  is the closest to  $1/2$ . Lines show the approximation (25). Dots and solid line are for even number of players, whereas squares and dashed line — odd.

$w(n, m)$ , namely that adding two citizens decreases the mafia winning-chance, whereas adding a citizen (which is an odd player) — increases.

The dynamics of the game is also of our interest. We calculate the probability that after  $t$  days and nights there is exactly a certain number of mafia members alive (13). We obtain closed-form expressions (14) and their convenient approximations (18).

In every realistic Mafia game, there is a psychological part. A dry mathematical model may be a backbone of a more complex one, but certainly, is not enough to describe a real game. During course of play, citizens gain some information — either by discovering another’s identity by themselves, or trying to catch messages. Furthermore, voting may be subject to some kind of witch-hunt mentality. Moreover, rarely are all players the same — usually there are ones with higher and lower influence on the others.

All these processes may be investigated mathematically, or numerically. There is still a lot of challenge in finding an appropriate model and obtaining the results. Comparison of the theory with the experiment may be crucial.

Out of author’s personal experience, in games when players don’t know each other, usually a random person is killed (e.g. a person who is always nervous, not just because of being in the mafia). When they know each other very well, very dim signals can be used to reveal one’s identity.

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### Appendix A: Solution of difference equation for $p_m(t)$

We want to solve the following set of difference equations (13),

$$p_m(t+1) = \frac{(N-2t)-m}{N-2t} p_m(t) + \frac{m+1}{N-2t} p_{m+1}(t) \quad : \quad m \in \mathbb{Z}$$

with the initial condition  $p_m(0) = \delta_{mM}$ . Let's introduce a generating function

$$F(t, z) = \sum_{m=0}^{\infty} p_m(t) z^m, \quad (\text{A1})$$

with the respective initial condition  $F(0, z) = z^M$ . After side-by-side multiplication of (13) by  $z^m$  and summation we get

$$\begin{aligned} F(t+1, z) &= F(t, z) - \frac{z}{N-2t} \frac{\partial F(t, z)}{\partial z} + \frac{1}{N-2t} \frac{\partial F(t, z)}{\partial z} \\ &= \underbrace{\left( 1 - \frac{z-1}{N-2t} \frac{\partial}{\partial z} \right)}_{\mathbf{B}(t)} F(t, z). \end{aligned} \quad (\text{A2})$$

The linear operator  $\mathbf{B}(t)$  has the eigenvectors of the form  $(z-1)^k$  with the corresponding eigenvalues  $(1 - k/(N-2t))$ . As the eigenvectors do not depend on  $t$ , the computation of  $F(t, z)$  is straightforward:

$$\begin{aligned} F(t, z) &= \mathbf{B}(t-1) \mathbf{B}(t-2) \cdots \mathbf{B}(0) z^M \\ &= \sum_{i=0}^M \binom{M}{i} (z-1)^i \left( 1 - \frac{i}{N-2(t-1)} \right) \left( 1 - \frac{i}{N-2(t-2)} \right) \cdots \left( 1 - \frac{i}{N} \right) \\ &= \sum_{k=0}^M \binom{M}{i} (z-1)^i \frac{(N-2t)!!}{N!!} \frac{(N-i)!!}{(N-2t-i)!!}. \end{aligned} \quad (\text{A3})$$

Applying once more the binomial theorem, and comparing the result (A3) to the definition of the generating function (A1) we get

$$p_m(t) = \sum_{i=m}^M \binom{M}{i} \binom{i}{m} (-1)^{i-m} \frac{(N-2t)!!}{N!!} \frac{(N-i)!!}{(N-2t-i)!!},$$

or (14).

### Appendix B: Solution of differential equation for $\tilde{p}_m(t)$

We want to solve the following set of differential equations (18),

$$\frac{d}{dt} \tilde{p}_m(t) = -\frac{m}{N-2t} \tilde{p}_m(t) + \frac{m+1}{N-2t} \tilde{p}_{m+1}(t) \quad : \quad m \in \mathbb{Z},$$

with the initial condition  $\tilde{p}(0) = \delta_{mM}$ . Once again, we employ a generating function

$$G(t, z) = \sum_{m=0}^{\infty} \tilde{p}_m(t) z^m, \quad (\text{B1})$$

with the respective initial condition  $G(0, z) = z^M$ . We get the partial differential equation

$$\frac{\partial G(t, z)}{\partial t} = \frac{(-z+1)}{N-2t} \frac{\partial G(t, z)}{\partial z}, \quad (\text{B2})$$

which we want to solve by the method of characteristics. Let's have integral curves of  $G(t, z)$  in the form of  $(t(\varphi), z(\varphi))$ ,

$$\frac{\partial G(t, z)}{\partial t} \frac{dt}{d\varphi} + \frac{\partial G(t, z)}{\partial z} \frac{dz}{d\varphi} = 0. \quad (\text{B3})$$

Comparison of (B2) with (B3) yields in

$$\frac{dt}{N - 2t} = d\varphi = \frac{dz}{z - 1}, \quad (\text{B4})$$

or

$$G(t, z) = f\left(\ln\left|\sqrt{N - 2t}(1 - z)\right|\right). \quad (\text{B5})$$

All we need is to find a function  $f(x)$  that satisfies the initial condition. The explicit form of the generating function reads

$$G(t, z) = \left(1 - \sqrt{1 - \frac{2t}{N}}(1 - z)\right)^M, \quad (\text{B6})$$

which gives the result

$$\tilde{p}_m(t) = \binom{M}{m} \left(1 - \sqrt{1 - \frac{2t}{N}}\right)^{M-m} \left(\sqrt{1 - \frac{2t}{N}}\right)^m$$

or (19).

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